MODIFIED GENERALIZED RIGHT ANGULAR DESIGNS

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Summary

A new class of PBIB designs with four associate-classes to be known as modified generalized rightangular designs is given. The use of these designs as nested factorial experiments is also discussed.

Introduction :

In this paper, the definition, analysis and application of a four associate class partially balanced incomplete block (PBIB) designs to be called Modified Generalized Rightangular (MGRA) designs for $v=lsp$, treatments are presented. The total $v-1$ degrees of freedom (*df*) for treatments have been partitioned into four orthogonal sets of $1-1$, $1(s-1)$, $1(p-1)$ and $1(p-1)(s-1)$ degrees of freedom, said to belong to the main effect A, main effect B within A, main effect C within A and the interaction BC within A respectively. The designs can be used as nested factorial experiments for three factors A, B, and C at l, s and p levels respectively. The Analysis is presented with this point of application in view. Several construction methods have also been presented.

For the definitions of the statistical terms used here, refer to Raghavarao [2].

Modified Generalized Rightangular Designs :

The modified generalized rightangular association scheme is defined as follows.

Definition 2.1. Let the $v=lsp$, l, s, p integers>1, treatments be denoted by the triplets $\alpha\beta\gamma$, $\alpha=1,2,..,l$, $\beta=1,2,...s$, $\gamma=1,2..,p$. Two treatments $\alpha\beta\gamma$ and $\alpha'\beta'\gamma'$ are

(*i*) First associates if $\alpha=\alpha'$, $\beta=\beta'$, $\gamma\neq\gamma'$,

(*ii*) second associates if $\alpha=\alpha'$, $\beta\neq\beta'$, $\gamma=\gamma'$,

(*iii*) third associates if $\alpha=\alpha'$, $\beta\neq\beta'$, $\gamma\neq\gamma'$

(*iv*) and fourths associates otherwise.

Clearly for this association scheme.

$$n_1=p-1,\ n_2=s-1;\ n_3=(p-1)(s-1);\ n_4=ps(l-1) \quad ...(2.2)$$

Let N be the incidence materix of a *MGRA* design, the latent roots θ_0, θ_1, θ_2, θ_3, and θ_4 of the NN' matrix with respective multiplicities α_0, α_1, α_2, α_3 and α_4 are,

$$\left.\begin{aligned}
&\theta_0=rk\\
&\theta_1=r-\lambda_4+(p-1)(\lambda_1-\lambda_4)+(s-1)(\lambda_2-\lambda_4)\\
&\qquad+(p-1)(s-1)(\lambda_3-\lambda_4)\\
&\theta_2=r-\lambda_2+(p-1)(\lambda_1-\lambda_3)\\
&\theta_3=r-\lambda_1+(s-1)(\lambda_1-\lambda_3)\\
&\theta_4=r-\lambda_1-(\lambda_2-\lambda_3)
\end{aligned}\right\} \quad ...(2.3)$$

$$\alpha_0=1,\ \alpha_1=l-1,\ \alpha_2=l(s-1),\ \alpha_3=l(p-1)$$
$$\alpha_4=l(p-1)s-1) \quad ...(2.4)$$

The latent roots $\phi_i(i=o,1,2,3,4)$ with respective multiplicities α_i of the C matrix given by

$$C=rI_v-\frac{1}{k}NN' \quad ...(2.5)$$

will be

$$\phi_i=r-\frac{\theta_i}{k}, \quad ...(2.6)$$

Analysis of MGRA designs as nested factorials :

Let y_{ijkq}, be the yield of the plot in the q-*th* block to which the

ijk-th treatment is allotted, be given by

$$y_{ijkq}=\mu+\beta_q+t_{ijk}+e_{ijkq} \qquad \ldots(3.1)$$

$$i=1,2\ldots l,\ j=1,2\ldots s,\ k=1,2,\ldots p, q=1,2,\ldots,b.$$

where μ is the general mean effect, β_q the effect of the q-th block, t_{ijk} the effect of the ijk-th treatment combination and e_{ijkq}'s are random errors supposed to be independently distributed with mean zero and constant variance σ^2, μ, β_q and $t_{ijk}'s$ are supposed to be fixed but unknown parameters.

Consider a factorial experiment lxsxp in three factors A, B and C where each of the factors B and C are nested within A and B and C are crossed. If t_{ijk} is the effect of the ijk-th treatment combination *i.e.*, when A is at the i-th level, B at j-th level and C at k-th level, $i=1,2,\ldots l$; $j=1,2,\ldots s$; $k=1,2,\ldots p$; then t_{ijk} can be written as

$$t_{ijk} = \alpha_i^A+\alpha_{ij}^B+\alpha_{ik}^C+\alpha_{ijk}^{BC} \qquad \ldots(3.2)$$

where

$$\alpha_i^A,\ \alpha_{ij}^B,\ \alpha_{ik}^C,\ \alpha_{ijk}^{BC}$$

are fixed effects due to the factors A,B,C and the interaction BC respectively.

and

$$\sum_i \alpha_i^A=\sum_j \alpha_{ij}^B=\sum_k \alpha_{ik}^C=\sum_j \alpha_{ijk}^{BC}=\sum_k \alpha_{ijk}^{BC}=0 \qquad \ldots(3.3)$$

for all i, j and k.

The estimates of the various effects are given by

$$\left.\begin{aligned}
\hat{\mu} &= \frac{y_{\cdots\cdot}}{lspr},\quad \hat{\alpha}_i^A=\frac{y_{i\cdots}}{spr}-\frac{y_{\cdots\cdot}}{lspr}\\
\hat{\alpha}_{ij}^B &= \frac{y_{ij\cdot\cdot}}{pr}-\frac{y_{i\cdots}}{spr}\\
\hat{\alpha}_{ik}^C &= \frac{y_{i\cdot k\cdot}}{sr}-\frac{y_{i\cdots}}{spr}\\
\hat{\alpha}_{ijk}^{BC} &= \frac{y_{ijk\cdot}}{r}-\frac{y_{ij\cdot\cdot}}{pr}-\frac{y_{i\cdot k\cdot}}{sr}+\frac{y_{i\cdots}}{spr}
\end{aligned}\right\} \qquad \ldots(3.4)$$

It can easily be shown that the sum of squares due to various effects are as follows

$$
\left.\begin{aligned}
&\text{sum of squares due to } A = \frac{\sum_i y^2_{i....}}{ispr} - \frac{y^2_{.....}}{lspr} \\
&\text{sum to squares due to } B \text{ within } A = \frac{\sum_i \sum_j y^2_{ij.}}{pr} - \frac{\sum_i y^2_{i....}}{spr} \\
&\text{sum of squares due to } C \text{ within } A = \sum_i \sum_k \frac{y^2_{i\,k.}}{sr} - \frac{\sum_i y^2_{i...}}{spr} \\
&\text{sum of squares due to } BC \text{ within } A = \sum_{i,} \sum_j \sum_k \frac{y^2_{ijk}}{r} - \sum_i \sum_j \frac{y^2_{ij..}}{pr} \\
&\qquad - \sum_i \sum_k \frac{y^2_{i.k.}}{sr} + \sum_i \frac{y^2_{i...}}{psr} \\
&\text{Block sum of squares} = \sum_q \frac{B_q^2}{k} - \frac{y^2_{....}}{lspr}
\end{aligned}\right\} \quad (3.5)
$$

where B_q is the q-th block total. Error sum of squares = by subtraction.

$$
\text{Total sum of squares} = \sum_i \sum_j \sum_k \sum_q y^2_{ijkq} - \frac{y^2_{....}}{lspr}.
$$

If $x_{i'}$, $y_{j'}$, $z_{k'}$, $w_{s'}$, $i'=1,2,...\alpha_1$, $j'=1, 2,...\alpha_2$ $k'=1,2,... \alpha_3$, $s'=1,2,...\alpha_4$ are the normalized characteristic vectors corresponding to the roots θ_1, θ_2, θ_3 and θ_4 of the NN' matrix, then it can easily be shown that

$$P_1=\sum_{i'=1}^{\alpha_1} x_{i'}x'_{i'}=s^{-1}p^{-1}E_{pp}\otimes I_1\otimes E_{ss}-s^{-1}p^{-1}1^{-1}E_{pp}\otimes E_{11}\otimes E_{ss}$$

$$P_2=\sum_{j'=1}^{\alpha_2} y_{j'}y'_{j'}=p^{-1}E_{pp}\otimes I_1\otimes I_s-p^{-1}s^{-1}E_{pp}\otimes I_1\otimes E_{ss}$$

$$P_3=\sum_{k'=1}^{\alpha_3} z_{k'}z'_{k'}=s^{-1}I_p\otimes I_1\otimes E_{ss}-s^{-1}p^{-1}E_{pp}\otimes I_1\otimes E_{ss} \qquad \ldots(3.6)$$

and $$P_4=\sum_{s'=1}^{\alpha_4} w_{s'}w'_{s'}=I_v-P_o-P_1-P_2-P_3$$

where $P_o=\frac{E_{vv}}{v}$

Following Aggarwal [1] it can be proved that the sum of squares due to A, sum of Squares due to B within A, sum of squares due to C within A and sum of squares due to BC within A are respectively

$$\frac{Q'P_1Q}{\phi_1},\quad \frac{Q'P_2Q}{\phi_2},\quad \frac{Q'P_3Q}{\phi_3} \text{ and } \frac{Q'P_4Q}{\phi_4} \qquad \ldots(3.7)$$

where Q is given by

$$C\hat{t}=Q. \qquad \ldots(3.8)$$

Following Scheffe [3]

Analysis of variance table can be given as below

Source of variation	*Degrees of Freedom*	*Sum of squares*
Blocks ignoring treatment	$b-1$	$\frac{1}{k}\sum_{q=1}^{b} B_q^2-\frac{y^2_{....}}{lspr}$
main effect A,	$(l-1)$	$\sum_i \frac{y^2_{i...}}{spr}-\frac{y^2_{....}}{lspr}=\frac{Q'P_1Q}{\phi_1}$

main effect B within A	$l(s-1)$	$\sum_i \sum_j \frac{y_{ij..}}{pr} - \sum_i \frac{y_{i...}}{spr} = \frac{Q'P_2Q}{\phi_2}$
main effect C ithin A	$l(p-1)$	$\sum_i \sum_k \frac{y^2_{i.k.}}{sr} - \sum_i \frac{y^2_{i...}}{spr} = \frac{QP_3Q}{\phi_3}$
Interaction BC within A	$l(s-1)(p-1)$	$\sum_i \sum_j \sum_k \frac{y^2_{ijk.}}{r} - \sum_i \sum_j \frac{y^2_{ij..}}{pr} - \sum_i \sum_k \frac{y^2_{i.k.}}{sr} + \sum_i \frac{y^2_{i...}}{psr} = \frac{Q'P_4Q}{\phi_4}$
error	$lspr-lsp-b+1$	By subtraction
Total	$lspr-1$	$\sum_i \sum_j \sum_k \sum_q y^2_{ijkq} - \frac{y^2_{....}}{lspr}$

An example of a *MGRA* design in which the main effect A is completely confounded but the other effects are unconfounded, is given.

Example 3.1. For, $v=12$, $l=2$, $s=2$, $p=3$ the following 12

Rep I	Rep II	Rep III
110	001 011 101 111	002 012 102 112
	010 000 110 100	010 000 110 100
	102	011 001 111

Suppose a *BIB* (p, r^*, λ) exists. Now we have the following theorems.

Theorem 4.1 Let N^* be the incidence matrix of a symmetrical $BIB(p, r^*, \lambda)$. In N^* if we replace unity by $\bar{A}_1$, the complement of A_1 and zero by A_2, the resulting matrix is the incidence matrix of a *MGRA* design with parameters

$$\left.\begin{array}{l} v = lsp = b,\ r = ps(l-1) + r^* = k \\ \lambda_1 = ps(l-1) + \lambda, \lambda_2 = ps(l-1) = \lambda_3, \lambda_4 = ps(l-2) + 2r^* \end{array}\right\} \quad \ldots(4.2)$$

Proof: Let N be the matrix obtained after replacing unity by $\bar{A}_1$ and zero by A_2 in N^*. Then

$$E_{1v}N = [ps(l-1) + r^*]E_{1b} \quad \ldots(4.3)$$

$$NE_{b1} = [ps(l-1) + r^*]E_{v1} \quad \ldots(4.4)$$

$$NN^l = I_p \otimes (P-Q) + E_{pp} \otimes Q \quad \ldots(4.5)$$

where

$$P = [ps(l-1) + r^*]A_o + ps(l-1)A_1 + [ps(l-2) + 2r^*]A_2$$

and

$$Q = [ps(l-1) +]A_o + ps(l-1)A_1 + [ps(l-2) + 2r^*]A_2.$$

Also

$$NN' = I_p \otimes (A-B) + E_{pp} \otimes B \quad \ldots$$

where $A = rA_0 + \lambda_2 A_1 + \lambda_4 A_2$

and $B = \lambda_1 A_0 + \lambda_3 A_1 + \lambda_4 A_2$

From (4.5) and (4.6) we get $\lambda_1, \lambda_2 \lambda_3$ and λ_4.

...e theorem.

Theorem 4.3 In the incidence matrix of a symmetrical $BIB(p,s^*,\lambda)$ if we replace unity by A_2 and zero by $\bar{A}_0$ then the resulting matrix is the incidence matrix of a *MGRA* design with parameters,

$$v=lsp=b, r=p(sl-1)-r^*(s-1)=k$$

$$\lambda_1=s(pl-2r^*+\lambda)-(p-2r^*+\lambda), \lambda_2=p(sl-2)-r^*(s-2)$$

$$\lambda_3=s(pl-2r^*+\lambda)-2(p-2r^*+\lambda), \lambda_4=p(sl-2)-2r^*(s-1).$$

Theorem 4.4 In the incidence matrix of a symmetrical $BIB(p,r^*,\lambda)$, if we replace unity by A_0 and zero by $\bar{A}_2$ then the resulting matrix is the incidence matrix of a *MGRA* design with parameters,

$$v=lsp=b, r=r^*+(p-r^*)s=k$$

$$\lambda_1=p+(p-2r^*+\lambda)(s-1), \lambda_2=(p-r^*)s$$

$$\lambda_3=(p-2r^*+\lambda)s+2(r^*-\lambda), \lambda_4=0$$

Theorem 4.5 If N^* is the incidence matrix of a symmetrical $BIB(p,r^*,\lambda)$ then $N^* \otimes A_1$ is the incidence matrix of a *MGRA* design with parameters,

$$v=lsp=b, r=r^*(s-1)=k,$$

$$\lambda_1=\lambda(s-1), \lambda_2=r^*(s-2), \lambda_3=\lambda(s-2), \lambda_4=0.$$

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